Wave function of the radion in the brane background with a massless scalar field and a self-tuning problem

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We consider flat solutions in a brane world model with a massless scalar field appearing in 5D H^2_{MNPQ}. Since those solutions have naked bulk singularities or are such that the 4D Planck mass is divergent, we should have a compact extra dimension, the size of which is then fixed by brane tension(s) and the bulk cosmological constant. By inspecting scalar perturbations around the flat solutions, we find that there is no zero mode for the radion and radion modes have only positive masses. This result confirms the fact that the radius of the extra dimension cannot be adjusted independently of the parameters of the model, and thus that the model considered does not solve the problems encountered in other approaches to the self-tuning of the cosmological constant.

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I. INTRODUCTION

The Randall-Sundrum (RS) models [1,2] were initially motivated by solving the hierarchy problem in a geometric way [1] and giving an alternative to the conventional Kaluza-Klein compactification of the extra dimension thanks to the presence of a normalizable zero mode of the graviton [2]. The RS solutions require one (or two) fine-tuning condition(s) between brane tension(s) and a bulk cosmological constant by consistency. That is to say, if the fine-tuning condition(s) was not satisfied, our universe should appear curved with nonzero 4D effective cosmological constant. Nonetheless, because of the fact that there exists a 4D flat solution in the RS models even with a nonzero brane tension(s) and a nonzero negative bulk cosmological constant, the cosmological constant problem appears in a new interesting perspective. Thus we search for an extra dimensional solution to the cosmological constant in the RS type models [3–14,16]. Here we require that there should exist a flat solution for a finite range of parameters and without any fine-tuning between input parameters, which will be called *self-tuning*. This idea can be realized by an integration constant appearing in the warp factor of the metric. That is to say, once a flat solution is obtained by determining the integration constant without a fine-tuning between given Lagrangian parameters, the integration constant is adjustable due to the dynamics of a bulk field to maintain the flatness of the metric for different sets of Lagrangian parameters of the finite range. The criteria for a consistent self-tuning solution with the extra dimension are the following: there should appear an integration constant to be determined by the boundary condition; there should be no naked singularities in the bulk space; and the 4D Planck mass should be finite.

In early attempts with a massless bulk scalar field coupled to the brane in a flat bulk [4,5], the self-tuning flat solution was obtained by using integration constants appearing in the warp factor and it was even claimed that there exists only a flat solution for the Z_2 symmetric bulk space with a special coupling of the scalar field to the brane [4,6]. However, since the extra dimension in such solutions should end with naked singularities in the bulk space to give a finite 4D Planck mass, a fine-tuning is indispensible in regularizing the naked singularity with another brane [7,8]. With a nonconventional kinetic term of $1/H^2$, where $H^2 = H_{MNPQ}H^{MNPQ}$, it has been shown that there exists a self-tuning solution [12].

On the other hand, there have appeared models where one introduces a massless bulk scalar field not coupled to the brane and a nonzero bulk cosmological constant [11,13,14]. In those attempts, the scalar field gives rise to an integration constant in the warp factor to be determined by the boundary condition because it contributes to the energy-momentum tensor in the Einstein equations.[1] In those attempts, there also appear naked singularities or the warp factor becomes divergent away from the brane as in the flat bulk case. Thus it is necessary to introduce another brane to cut off the extra dimension in the same way as in the case where the brane is coupled to the scalar field. In this case, however, the extra dimension should be cut off before the naked singularities, unlike in the case where the brane is coupled to the scalar field.[2]

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[1]The integration constant from the scalar field itself is irrelevant to self-tuning because it is just contained in the warp factor as a *multiplicative* factor.

[2]Locating another brane at the singularity would give rise to an *infinite* brane tension, unlike the case with a brane coupling.

When the scalar field does not couple to the brane, another brane introduces a parameter, the distance between two branes.[3] In this paper, we present flat solutions with a static extra dimension in which the bulk and brane cosmological constants do not need to be fine-tuned, but those rather determine the size of the extra dimension in terms of those parameters by the boundary conditions at the branes. Moreover, for a positive bulk cosmological constant, it may also be possible to compactify the extra dimension even without the need to introduce another brane by identifying the two extrema of the warp factor symmetric around the $y=0$ brane [14]. In this case with a single brane, there is no direct fine-tuning between bulk and brane cosmological constants either, but the radius of the extra dimension is also fixed in terms of those parameters. However, once the radius is determined by the input parameters, the dilatation symmetry of the extra dimension is broken. Thus, we expect that the dilatation moduli of the radius, which is called the radion, does not have a massless mode. In this case, the radius cannot be changed to maintain the flat solution for a small change of brane tension.

In this paper, we make a consistency check for our flat solutions with a constant radion field: (1) each flat solution is an extremum of the radion effective potential and (2) each flat solution is a global minimum of the radion effective potential. First we obtain the equation of motion for the radion field and show that the determined radius is an extremum of the radion effective potential. Then we also study scalar and radion perturbations around our flat solutions with arbitrary bulk cosmological constant. As expected, we find that there is no massless mode of the radion because of the boundary condition coming from the need to cut off the extra dimension with or without another brane. Moreover, the radion has only a positive mass, which implies that each flat solution with the fixed radius is a minimum of the radion potential. Concerning the self-tuning problem in our model, we can address the problem that, due to the absence of a massless radion, the radius of the extra dimension is not adjustable for self-tuning the cosmological constant. Thus, even if there is no direct relation between bulk and brane cosmological constants in our case, one fine-tuning condition cannot be avoided to maintain the flat solution for a small change of brane tension.

For completeness, we use the gauge transformations preserving the scalar perturbations to find that there exist a massless graviscalar and a massless spin-2 graviton while there is no massless vector mode. The massive excitations of the spin-2 graviton are shown to be positive, which guarantees the stability of the flat solutions together with the positive radion mass.

This paper is organized as follows. In the next section, we provide the model setup for consideration and present the flat solutions. In Sec. III, we perturb the scalar field around the flat solutions, identify the radion spectrum, and discuss the self-tuning problem in our model. Then, in Sec. IV, we present the effective action of the radion. In Sec. V, we deal with the graviton perturbations for completeness. Section VI is a conclusion.

[3]Note that the radius of the extra dimension is not determined in the Randall-Sundrum model.

II. MODEL SETUP

On top of the RS model [1,2], we introduce a three-form field in the bulk without coupling to the brane. Because we will cut off the extra dimension, we include the sum of brane actions. Then, the 5D action of our model setup is

$$S=\int d^4x\int dy\sqrt{-g}\left(\frac{M^3}{2}R-\Lambda_b -\frac{1}{2\cdot 4!}H_{MNPQ}H^{MNPQ}+\sum_i \mathcal{L}_m^{(i)}\delta(y-y_i)\right). \quad (1)$$

In order to get 4D flat solutions, let us take the ansatz for the metric as

$$ds^2=\beta^2(y)\,\eta_{\mu\nu}dx^\mu dx^\nu+dy^2 \quad (2)$$

where $(\eta_{\mu\nu})=\mathrm{diag}(-1,+1,+1,+1)$. Then the components of the Einstein tensor are

$$G_{\mu\nu}=g_{\mu\nu}\left[3\left(\frac{\beta'}{\beta}\right)^2+3\left(\frac{\beta''}{\beta}\right)\right],$$

$$G_{55}=6\left(\frac{\beta'}{\beta}\right)^2, \quad (3)$$

where a prime denotes differentiation with respect to y. With the brane tensions Λ_1 and Λ_2 at the $y=0$ and $y=y_c$ branes, respectively, and the bulk cosmological constant Λ_b, the energy momentum tensors are

$$T_{MN}=-g_{MN}\Lambda_b -\frac{\sqrt{-g^{(4)}}}{\sqrt{-g}}g_{\mu\nu}\delta^\mu_M\delta^\nu_N\sum_i \Lambda_i\delta(y-y_i)+\tilde{T}_{MN}, \quad (4)$$

$$\tilde{T}_{MN}=\frac{1}{4!}\left(4H_{MPQR}H_N^{PQR}-\frac{1}{2}H^2g_{MN}\right) \quad (5)$$

$$=\nabla_M\phi\nabla_N\phi-\frac{1}{2}g_{MN}(\nabla\phi)^2, \quad (6)$$

where we used the fact that a three-form field in 5D spacetime is dual to a pseudoscalar field as $H_{MNPQ}=\sqrt{-g}\epsilon^R_{MNPQ}\nabla_R\phi$. Below, we imply "pseudoscalar" when we write "scalar" without any confusion because our spin-0 field is only the pseudoscalar from H_{MNPQ}. Here, when the dual relation is inserted in the kinetic term of the action, its overall sign is the opposite of that in the case of a scalar field. However, there does not arise such an inconsistency between the action and the Einstein equation when the surface term is taken into account that arises for the well-defined variation of the action with respect to the three-form field.

We also take the ansatz for the nonvanishing components of the four-form field $H_{\mu\nu\rho\sigma}$ as

$$H_{\mu\nu\rho\sigma}=\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}f(y) \tag{7}$$

where $\mu, \ldots$ run over the Minkowski indices 0, 1, 2, and 3. The Bianchi identity $dH=0$ implies $f^2=2A/\beta^8$ in the bulk with a positive constant A. With the above ansatz, the field equation for the four-form field is satisfied:

$$\partial_M[\sqrt{-g}H^{MNPQ}]=0. \tag{8}$$

The two relevant Einstein equations, the (55) and $(\mu\mu)$ components, read as follows:

$$6\left(\frac{\beta'}{\beta}\right)^2=-\Lambda_b+\frac{A}{\beta^8}, \tag{9}$$

$$3\left(\frac{\beta'}{\beta}\right)^2+3\left(\frac{\beta''}{\beta}\right)=-\Lambda_b-\Lambda_1\delta(y)-\Lambda_2\delta(y-y_c)-\frac{A}{\beta^8}. \tag{10}$$

The solutions of Eqs. (9) and (10) with Z_2 symmetry are, [13]

for $\Lambda_b<0$,

$$(1)\,\beta(|y|)=\left(\frac{a}{k}\right)^{1/4}[\sinh(-4k|y|+c)]^{1/4}, \tag{11}$$

$$(2)\,\beta(|y|)=\left(\frac{a}{k}\right)^{1/4}[\sinh(4k|y|+c)]^{1/4}, \tag{12}$$

for $\Lambda_b>0$,

$$(1)\,\beta(|y|)=\left(\frac{a}{k}\right)^{1/4}[\sin(-4k|y|+c)]^{1/4}, \tag{13}$$

$$(2)\,\beta(|y|)=\left(\frac{a}{k}\right)^{1/4}[\cos(-4k|y|+c)]^{1/4}, \tag{14}$$

for $\Lambda_b=0$,

$$(1)\,\beta(|y|)=(-4a|y|+c)^{1/4}, \tag{15}$$

$$(2)\,\beta(|y|)=(4a|y|+c)^{1/4}, \tag{16}$$

where $k\equiv\sqrt{|\Lambda_b|/6}$ and a is defined in terms of A,

$$a\equiv\sqrt{\frac{A}{6}}\,. \tag{17}$$

We note that the β's of Eqs. (12) and (16) do not give localized gravity on the $y=0$ brane while the β's of Eqs. (11), (13), and (15) have naked singularities at $|y|=c/(4k)$ or $|y|=c/(4a)$, and β of Eq. (14) has at $|y|=(c+\pi/2)/(4k)$. Therefore, to get effective four-dimensional gravity or to avoid singularities in the bulk, it is indispensable to cut the extra dimension such that it has a finite length. This can be done by introducing another brane.

Then, since the Z_2 symmetry and the periodicity for the compact dimension give rise to the boundary conditions at the branes,

$$\left.\frac{\beta'}{\beta}\right|_{y=y_i^+}\equiv-\frac{\Lambda_i}{6}, \tag{18}$$

consistency requires the following relations for the above three cases:

$$\text{for}\quad \Lambda_b<0,\quad \pm c=\coth^{-1}\left(\frac{k_1}{k}\right)=4ky_c-\coth^{-1}\left(\frac{k_2}{k}\right), \tag{19}$$

$$\text{for}\quad \Lambda_b>0,\quad (1)\,c=\cot^{-1}\left(\frac{k_1}{k}\right)=4ky_c-\cot^{-1}\left(\frac{k_2}{k}\right), \tag{20}$$

$$(2)\,c=-\tan^{-1}\left(\frac{k_1}{k}\right)=4ky_c+\tan^{-1}\left(\frac{k_2}{k}\right), \tag{21}$$

$$\text{for}\quad \Lambda_b=0,\quad \pm c=\frac{a}{k_1}=a\left(4y_c-\frac{1}{k_2}\right), \tag{22}$$

where $c>0$ for $\Lambda_b\leqslant 0$, $0<c<\pi/2$ for $\Lambda_b>0$, and

$$k_1\equiv\frac{\Lambda_1}{6},\quad k_2\equiv\frac{\Lambda_2}{6}. \tag{23}$$

For all the above cases, direct fine-tuning conditions appear between brane tensions, unlike in the RS case, but the size of the extra dimension is determined by the brane tensions and the bulk cosmological constant. In particular, for the $\Lambda_b>0$ case [Eq. (14)], as recently argued in Ref. [14], it may be possible to compactify the extra dimension without the need to introduce another brane by identifying the two extrema of the warp factor symmetric around the $y=0$ brane. In Fig. 1 we show the schematic behavior of the warp factor. If point Q corresponds to the location of another brane, then one has to introduce a brane tension there. However, if point P is chosen, it is not necessary to introduce another brane. That means the extra dimensional size depends on point P, which is determined by the other parameters in the theory. For this to be the case, we must have a shift symmetry in the radion direction whose distance is determined by the parameters in the theory; namely, we only have to take the boundary condition as the first one in Eq. (21) and the length size of the extra dimension can be regarded as being determined by the brane tension as $y_c=c/(4k)=-(4k)^{-1}\tan^{-1}(k_1/k)$ for $k_1<0$.

Now we have obtained flat solutions with zero effective cosmological constant on the brane with a static extra dimension, i.e., $g_{55}=1$. However, we should check (1) whether these solutions actually extremize the radion effective potential and, if so, (2) whether this extremum is a minimum or a maximum of the effective potential. Let us close this section

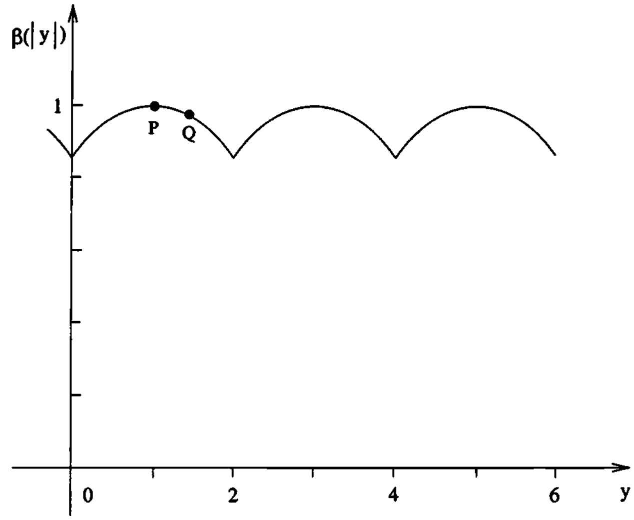

FIG. 1. The schematic behavior of the warp factor for $\Lambda_b > 0$. Point P is the case of our main interest.

with the first consistency test of our flat solutions and postpone the second one to the next section.

It was shown that the 5D extremization constraint can be a consistency check for any solutions with a constant radion field [15]. For the general metric ansatz with homogeneous and isotropic three-space given by

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)\delta_{ij}dx^i dx^j + b^2(t,y)dy^2, \tag{24}$$

we consider a linear combination of all the components of the 5D energy-momentum tensor from Einstein's equations,[4]

$$T^\mu_\mu - 2T^5_5 = -\frac{3}{n^2}\left(\frac{3\dot{a}\dot{b}}{ab} - \frac{\dot{b}\dot{n}}{bn} + \frac{\ddot{b}}{b}\right) + \frac{3}{b^2}\left(\frac{3a''}{a} + \frac{n''}{n} - \frac{3a'b'}{ab} + \frac{b'n'}{bn}\right) \tag{25}$$

where dot and prime denote differentiations with respect to t and y coordinates, respectively. Then, the above constraint can be regarded as the equation of motion of the time-dependent scalar field $b(t,y)$ by rewriting it as

$$\frac{1}{b}\nabla^\mu\nabla_\mu b = \frac{1}{3}(T^\mu_\mu - 2T^5_5) - \frac{3}{a}\left(\frac{a'}{b^2}\right)' - \frac{1}{n}\left(\frac{n'}{b^2}\right)'. \tag{26}$$

For solutions with a nonstatic extra dimension, the right-hand side (RHS) of the above equation vanishes when an extremum is reached. This is also the case with a static extra dimension because a solution with a constant radion field is defined as an extremum of the radion effective potential. Actually, we obtain the RHS with $b=1$ and $n=a=\beta(y)$,

$$\frac{2}{3}\left(-\Lambda_b - 3\frac{A}{\beta^8} - 6\left.\frac{\beta''}{\beta}\right|_{y\neq y_i}\right) - 8\delta(y-y_i)\left(\left.\frac{\beta'}{\beta}\right|_{y=y_i} + \frac{\Lambda_i}{6}\right). \tag{27}$$

Therefore, we can show that flat solutions with a static extra dimension satisfy the extremization constraint by using Eqs. (9), (10), and (18). In this extremization procedure, the size of the extra dimension is determined by the boundary conditions. This is different from the case with the RS flat solution, where the fine-tuning of the parameters leads to the extremization of the radion potential without determining the size of the extra dimension.

III. SCALAR PERTURBATIONS AND RADION

In order to determine whether the flat solutions with a static extra dimension obtained in the previous section are minima or a maxima of the radion effective potential, we should study scalar perturbations around the flat solutions. Then, the radion mode corresponds to altering the distance between the branes at the orbifold fixed points or to the size of the extra dimension, which appears as a scalar field in the effective 4D theory.

In the RS case without radius stabilization, there exists a massless radion for the flat solution since the size of the extra dimension is not determined [1,17,18]. And it is also shown that the radion has a positive mass for the 4D AdS solution while it has a negative mass for the 4D dS solution [19–21]. For the stabilized RS model, the radion mass is also investigated in detail in Refs. [22,23]. In models with metastable graviton and multigravity scenarios, it is shown that the radion dynamics is crucial to test their stability [24,25].

As we mentioned in the previous section, we regard the three-form field as a scalar field by duality on investigating perturbations for convenience. In that case, we only have to deal with the energy-momentum tensor coming from the scalar field as in Eq. (6) and a scalar field equation such as

$$\partial_M(\sqrt{-g}\,\partial^M\phi) = 0. \tag{28}$$

We thus take a general ansatz for the metric perturbations as

$$ds^2 = K^{-2}(z)[\eta_{MN} + h_{MN}(x,z)]dx^M dx^N, \tag{29}$$

$$\phi(x,z) = \phi_0(z) + \varphi(x,z), \tag{30}$$

where $z = \int^z dy/\beta(y)$, $K(z) = 1/\beta(y(z))$, and $(\phi_0')^2 = AK^6$.

Then, the linearized Einstein's tensor, the linearized energy-momentum tensor, and the linearized scalar equation read

[4]See, for instance, Eq. (75) in Ref. [13].

$$\delta G_{MN}=-\frac{\Box}{2}\overline{h}_{MN}+\partial_{(M}\partial^{P}\overline{h}_{N)P}-\frac{1}{2}\eta_{MN}\partial^{P}\partial^{Q}\overline{h}_{PQ} - \frac{3K'}{2K}(\partial_M h_{N5}+\partial_N h_{M5}-\partial_5 h_{MN})-3\eta_{MN} \times\left[\left(-\frac{K''}{K}+2\frac{K'^2}{K^2}\right)h_{55}-\frac{K'}{K}\partial^{P}\overline{h}_{P5}\right] -3\left[\frac{K''}{K}-2\frac{K'^2}{K^2}\right]h_{MN}, \tag{31}$$

$$\delta T_{\mu\nu}=\left(-3\frac{K''}{K}+6\frac{K'^2}{K^2}\right)h_{\mu\nu}+\sum_i \delta(z-z_i)\frac{\Lambda_i}{2K}h_{55}\eta_{\mu\nu} +\frac{1}{2}(\phi_0')^2 h_{55}\eta_{\mu\nu}-\phi_0'\varphi'\eta_{\mu\nu}, \tag{32}$$

$$\delta T_{\mu 5}=6\frac{K'^2}{K^2}h_{\mu5}+\phi_0'\partial_\mu\varphi, \tag{33}$$

$$\delta T_{55}=6\frac{K'^2}{K^2}h_{55}-\frac{1}{2}(\phi_0')^2 h_{55}+\phi_0'\varphi', \tag{34}$$

$$\Box\varphi-3\frac{K'}{K}\varphi'+\frac{1}{2}\phi_0'(h_\mu^{\mu\prime}-h_{55}')=0, \tag{35}$$

where (M,N) is one-half of the symmetric combination, the prime denotes the derivative with respect to z, and $\partial^M \equiv \eta^{MN}\partial_N$, $\Box\equiv\eta^{\mu\nu}\partial_\mu\partial_\nu+\partial_z^2$. $\overline{h}_{MN}$ is defined as $\overline{h}_{MN} \equiv h_{MN}-\eta_{MN}h/2$ and h is the trace for h_{MN}.

Since we are interested in the scalar perturbations for investigating the radion, let us take the gauge choice in the metric ansatz (29) as

$$h_{\mu\nu}(x,z)=F(x,z)\eta_{\mu\nu}, \quad h_{\mu5}=0, \qquad h_{55}(x,z)=G(x,z). \tag{36}$$

Thus, the $(\mu\nu)$ component of the linearized Einstein tensor is written in the form

$$\delta G_{\mu\nu}=\partial_\mu\partial_\nu\left(-F-\frac{1}{2}G\right)+\cdots \tag{37}$$

where the ellipses all contain terms proportional to $\eta_{\mu\nu}$. The linear perturbations from the energy momentum tensor are also $\sim\eta_{\mu\nu}$. Therefore, we obtain an immediate relation such as

$$G=-2F. \tag{38}$$

Then the linearized $(\mu\mu)$, $(\mu5)$, and (55) Einstein's equations and the linearized scalar field equation are

$$\frac{3}{2}F''-\frac{15}{2}\frac{K'}{K}F'+6\left[2\left(\frac{K'}{K}\right)^2-\frac{K''}{K}\right] \times F+\left[\sum_i \frac{\Lambda_i}{K}\delta(z-z_i)+(\phi_0')^2\right]F+\phi_0'\varphi'=0, \tag{39}$$

$$-\frac{3}{2}\partial_\mu F'-3\frac{K'}{K}\partial_\mu F=\phi_0'\partial_\mu\varphi, \tag{40}$$

$$\frac{3}{2}\partial^\mu\partial_\mu F-6\frac{K'}{K}F'+\left[12\left(\frac{K'}{K}\right)^2-(\phi_0')^2\right]F-\phi_0'\varphi'=0, \tag{41}$$

$$\partial^\mu\partial_\mu\varphi+\varphi''-3\frac{K'}{K}\varphi'+3\phi_0'F'=0. \tag{42}$$

The $(\mu5)$ component Eq. (40) can be integrated to give

$$\phi_0'\varphi=-\frac{3}{2}\left(F'-2\frac{K'}{K}F\right)+f(z). \tag{43}$$

The metric ansatz (36) and Eqs. (38) and (43) with $f(z)=0$ fix our gauge choice. Eliminating $\phi_0'\varphi'$ from Eqs. (39) and (41) gives rise to an equation for F only as

$$\partial^\mu\partial_\mu F+F''-9\frac{K'}{K}F' +\left[16\left(\frac{K'}{K}\right)^2-4\frac{K''}{K}+\sum_i\frac{2\Lambda_i}{3K}\delta(z-z_i)\right]F=0 \tag{44}$$

from which we can easily obtain the boundary conditions at the branes,

$$\left[F'-2\frac{K'}{K}F\right]_{z=z_i^+}=0, \tag{45}$$

where we used $K''/K=2AK^6/3+(\Lambda_i/3K)\delta(z-z_i)$ and $\Lambda_i=6K'|_{z=z_i}$. It is straightforward to check the scalar field equation (42) by multiplying Eq. (42) by ϕ_0' and using Eqs. (43), (41), and the background equations (9),(10) in the z coordinate. Thus, F must satisfy Eq. (45) and the bulk equation

$$\partial^\mu\partial_\mu F+F''-9\frac{K'}{K}F' +\left[16\left(\frac{K'}{K}\right)^2-\frac{4K''}{K}\right]F=0. \tag{46}$$

Note that studying F is equivalent to studying the radion, in view of Eqs. (36) and (38).

To make a Kaluza-Klein reduction of the radion field to 4D, we choose a separation of variables as $F(x,z)=\psi(z)\rho(x)$. Taking a rescaling such as $\tilde{\psi}=\psi/(\phi_0'K^{3/2})$ [26], we obtain the 4D equation of the radion field and the bulk equation for the wave function of the radion

$$(\partial^\mu \partial_\mu - m^2)\rho(x) = 0, \tag{47}$$

$$[-\partial_z^2 + V(z)]\tilde{\psi}(z) = m^2 \tilde{\psi}(z), \tag{48}$$

where

$$V(z) = \xi \left(\frac{1}{\xi}\right)'', \tag{49}$$

$$\xi = \frac{\phi_0'}{K^{1/2}K'}. \tag{50}$$

The above equation corresponds to nothing but a supersymmetric quantum mechanics,

$$Q^\dagger Q \tilde{\psi}(z) \equiv \left[\partial_z + \xi\left(\frac{1}{\xi}\right)'\right]\left[-\partial_z + \xi\left(\frac{1}{\xi}\right)'\right]\tilde{\psi}(z) = m^2 \tilde{\psi}(z). \tag{51}$$

The Hermiticity of the above differential operator guarantees the positivity of the mass spectrum with $m^2 \geqslant 0$: there is no tachyonic mode of radion. The bulk solution for the massless radion field with $m^2 = 0$ is given by a linear combination such as

$$\tilde{\psi}_0(z) = \frac{1}{\xi}\left(c_0 + d_0 \int_0^z dz\, \xi^2\right) \tag{52}$$

or

$$\psi_0(z) = K^2 K' \left[c_0 + d_0 \int_0^z dz \left(\frac{\phi_0'}{K^{1/2}K'}\right)^2\right] \tag{53}$$

where c_0 and d_0 are integration constants. Then, we can take the wave function of the massless radion to be consistent with the Z_2 symmetry,

$$\psi_0(z) = \begin{cases} K^2 K' \left[c_0 + d_0 \int_0^z dz \left(\frac{\phi_0'}{K^{1/2}K'}\right)^2\right] \\ \qquad \text{for} \quad z > 0, \\ -K^2 K' \left[c_0 - d_0 \int_0^z dz \left(\frac{\phi_0'}{K^{1/2}K'}\right)^2\right] \\ \qquad \text{for} \quad z < 0. \end{cases} \tag{54}$$

Then, using the boundary condition at the $z=0$ (or $y=0$) brane from Eq. (45),

$$\left[\psi' - 2\frac{K'}{K}\psi\right]_{z=0^+} = 0, \tag{55}$$

we obtain the ratio between the two integration constants as

$$(\mathrm{A}):\quad \frac{c_0}{d_0} = -\left[\frac{(\phi_0')^2}{K''K'K}\right]_{z=0^+}. \tag{56}$$

On the other hand, since the extra dimension should be cut off with or without another brane to escape a bulk singularity or a divergent 4D Planck mass, an additional boundary condition appears. For the case without another brane, the derivative of the metric perturbation should be zero at the end of the extra dimension, as for the background metric. Therefore, in view of Eq. (45), another boundary condition should be the same irrespective of the existence of an additional brane as

$$(K^{-2}\psi)'|_{z=z_c^-} = 0. \tag{57}$$

Thus we also find another necessary condition for integration constants:

$$(\mathrm{B}):\quad \frac{c_0}{d_0} = -\left[\frac{(\phi_0')^2}{K''K'K}\right]_{z=z_c^-} - \int_0^{z_c} dz \left(\frac{\phi_0'}{K^{1/2}K'}\right)^2. \tag{58}$$

However, the two equations (56) and (58) cannot be satisfied simultaneously because the difference (A)-(B) between the two turns out to be nonzero,

$$(A) - (B) = -3\int_0^{z_c} dz K^{-3} \neq 0. \tag{59}$$

As a result, *the massless mode of radion should be regarded as being projected out* by another boundary condition appearing in cutting off the extra dimension, irrespective of whether the extra dimension is compactified with one or two branes. (The absence of a massless radion was also shown in Ref. [22] in the existence of the general Goldberger-Wise scalar potential. But our case without any scalar potential was not covered there.) Thus we have confirmed that *the radius of the extra dimension cannot be changed independently of the cosmological constants* by showing explicitly that there is no massless mode of the radion. Then one cannot escape one fine-tuning condition [13] to maintain the flat solution with fixed radius for small changes of input parameters. Moreover, since the radion gets only a positive mass from Eq. (51), we find that each flat solution of Einstein's equations [Eqs. (11)–(16)] is stable under scalar perturbations.

IV. EFFECTIVE ACTION OF THE RADION

We would like now to discuss the effective theory for the massive modes of the radion with matter localized on the brane. So let us substitute the perturbed metric (36) into the linearized 5D action and integrate out the extra dimension. Since the massive radion couples to the energy-momentum tensor localized on the brane(s), its effective action is derived as

$$S_{eff}=\int d^4x dz \sqrt{-g^{(4)}}K^{-5}\Bigg[\left(\frac{M^3}{2}\right)\frac{1}{2}h^{MN}\delta \times\left(R_{MN}-\frac{1}{2}g_{MN}R-M^{-3}T_{MN}\right) -\frac{1}{2}\sum_i Kh^{\mu\nu}T^{(i)}_{\mu\nu}\delta(z-z_i)\Bigg] \tag{60}$$

$$=\int d^4x\sqrt{-g^{(4)}}\Bigg[\frac{M^3}{2}\int_{-z_c}^{z_c}dzK^{-3}\left(\frac{3}{2}F(\Box-m^2)F\right) +\frac{1}{2}\sum_i (K^{-2}F)\Big|_{z_i}\eta^{\mu\nu}T^{(i)}_{\mu\nu}\Bigg]. \tag{61}$$

Here we make a separation of variables such as $F(x,z)=\psi(z)\rho(x)$ and rescale the radion field such that its kinetic term has a canonical form. Then, the resultant effective action is the following:

$$S_{eff}=\int d^4x\sqrt{-g^{(4)}}\Bigg[\frac{1}{2}\tilde{\rho}(x)(\Box-m^2)\tilde{\rho}(x) +\sum_i \frac{1}{2M^{(i)}_{rad}}\tilde{\rho}(x)T^{\mu(i)}_{\mu}\Bigg] \tag{62}$$

where the coupling of the radion to matter localized at each brane is identified as

$$\frac{1}{M^{(i)}_{rad}}=\left[\frac{3M^3}{2}\int_0^{z_c}dzK^{-3}\psi^2\right]^{-1/2}(K^{-2}\psi)|_{z_i}. \tag{63}$$

V. GRAVITON PERTURBATIONS

In this section, for completeness, on top of the scalar perturbations corresponding to the radion in the previous section, let us consider the graviton perturbations with the general metric ansatz

$$h_{\mu\nu}(x,z)=F(x,z)\eta_{\mu\nu}+\tilde{h}_{\mu\nu}(x,z),$$

$$h_{\mu5}\neq0,\quad h_{55}(x,z)=G(x,z). \tag{64}$$

Then, with the gauge conditions (38) and (43) for the scalar perturbations, we use a 4D gauge transformations to impose the de Donder gauge condition for the graviton

$$0=\partial^\mu\bar{\tilde{h}}_{\mu\nu}=\partial^\mu\overline{h}_{\mu\nu} \tag{65}$$

where $\overline{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}h^\lambda_\lambda\eta_{\mu\nu}$ and the second equality comes from Eq. (38). Now we have fixed five of the 15 degrees of freedom of the 5D metric perturbations.

Assuming that the equations for scalar perturbations (39)–(42) are reproduced for the general gauge choice with Eqs. (64) and (65), we obtain the equations for the graviton from Eqs. (31)–(34) as

$$-\frac{1}{2}\left(\Box-3\frac{K'}{K'}\partial_5\right)\tilde{h}_{\mu\nu}+\frac{1}{4}\eta_{\mu\nu}\left(\Box+\partial_5^2-6\frac{K'}{K}\partial_5\right)\tilde{h}^\mu_\mu +\left(\partial_5-3\frac{K'}{K}\right)(\partial_{(\mu}h_{\nu)5}-\eta_{\mu\nu}\partial^\lambda h_{\lambda5})=0, \tag{66}$$

$$-\frac{1}{2}\left(\partial^\lambda\partial_\lambda+6\frac{K''}{K}\right)h_{\mu5}+\frac{1}{2}\partial_\mu\left(\partial^\lambda h_{5\lambda}-\frac{1}{2}\partial_5\tilde{h}^\lambda_\lambda\right)=0, \tag{67}$$

$$\frac{1}{4}\partial^\lambda\partial_\lambda\tilde{h}^\mu_\mu+\frac{3K'}{K}\left(\partial^\lambda h_{\lambda5}-\frac{1}{2}\partial_5\tilde{h}^\mu_\mu\right)=0. \tag{68}$$

Then we find that the traces of both Eq. (66) and Eq. (68) can be satisfied simultaneously with

$$\partial^\lambda h_{5\lambda}-\frac{1}{2}\partial_5\tilde{h}^\mu_\mu=0, \tag{69}$$

under which Eqs. (68) and (67) become, respectively,

$$\partial^\lambda\partial_\lambda\tilde{h}^\mu_\mu=0, \tag{70}$$

$$\left(\partial^\lambda\partial_\lambda+6\frac{K''}{K}\right)h_{\mu5}=0. \tag{71}$$

Therefore, there appears a massless graviscalar field $\tilde{h}^\mu_\mu$, which couples to the trace of the 4D energy momentum tensor as for the radion. On the other hand, since Eq. (71) cannot be satisfied either on the brane or in the bulk, $h_{\mu5}=0$, i.e., there is no massless vector mode, which is consistent with Eq. (69) since $(\tilde{h}^\mu_\mu)'=0$ from comparing the linearized scalar equations (35) and (42).

From Eq. (66), we also observe that the transverse traceless spin-2 graviton ($\tilde{h}^{TT}_{\mu\nu}$) is automatically decoupled due to $\tilde{h}^{\mu\prime}_\mu=h_{\mu5}=0$ and the de Donder gauge (65):

$$\left(\Box-3\frac{K'}{K}\partial_5\right)\tilde{h}^{TT}_{\mu\nu}=0. \tag{72}$$

Consequently, it turns out that there exists a massless spin-2 graviton since the above equation is satisfied by $\tilde{h}^{TT}_{\mu\nu}(x,z)=c_0e^{ipx}\epsilon_{\mu\nu}$ with $p^2=0$, where c_0 is a constant and $\epsilon_{\mu\nu}$ is a polarization tensor.

With a separation of variables as $\tilde{h}^{TT}_{\mu\nu}=K^{3/2}(z)\tilde{\psi}(z)e^{ipx}\epsilon_{\mu\nu}$ $(p^2=-m^2)$, we obtain a Schrödinger-like equation again [27]:

$$(-\partial_z^2+V(z))\tilde{\psi}(z)=m^2\tilde{\psi}(z) \tag{73}$$

where

$$V(z)=\frac{15}{4}\left(\frac{K'}{K}\right)^2-\frac{3K''}{2K}. \tag{74}$$

Therefore, we find that $m^2 \geqslant 0$, i.e., there is no tachyonic state of the graviton since the above equation can be regarded as a supersymmetric quantum mechanics,

$$Q^\dagger Q \tilde{\psi}(z) \equiv \left(\partial_z - \frac{3}{2}\frac{K'}{K} \right) \left(-\partial_z - \frac{3}{2}\frac{K'}{K} \right) \tilde{\psi}(z) = m^2 \tilde{\psi}(z). \tag{75}$$

VI. CONCLUSION

In this paper, by introducing a 5D massless scalar field not coupled to the brane(s), we found that there exist flat solutions with no direct fine-tuning relations between brane tension(s) and a bulk cosmological constant. Since those solutions have a naked singularity or the 4D Planck mass becomes divergent for a noncompact extra dimension, the extra dimension should be compactified with or without another brane. Thus there appear two boundary conditions one at each end of the extra dimension. One boundary condition at the $y=0$ brane determines the integration constant of the warp factor and the other one fixes the radius of the extra dimension in terms of brane and bulk cosmological constants. At first sight, the radius seems to be a self-tuning parameter, like an integration constant. However, it is expected that the fixed radius breaks the dilatation symmetry of the radius such that the radius cannot be changed independently of the parameters in the model to maintain the flat solution.

We showed that the fixed radius is an extremum of the radion effective potential from the equation of motion for the radion field. On investigating scalar and radion perturbations around the flat solutions, we also found that the massless mode of the radion is projected out by another boundary condition appearing from the procedure of cutting off the extra dimension. Thus we confirmed that the dilatation symmetry of the radius is broken so that the radius of the extra dimension cannot be adjusted independently of the cosmological constants. Moreover, the radion has only a positive mass, which implies that each flat solution with a static radius is a minimum of the radion potential.

For completeness, we also found that there exist a massless graviscalar and a massless spin-2 graviton while there is no massless vector mode. The spectrum of massive spin-2 graviton is shown to be semipositive, which assures the stability of the flat solutions in our model together with the positive spectrum of the radion.

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